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Energetics of oscillating lifting surfaces by the use of integral conservation laws

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The energetics of oscillating lifting surfaces in two and three dimensions is calculated by the use of integral conservation laws in inviscid incompressible flow for general and harmonic transverse oscillations. Wing deformations are prescribed as a function of time and total thrust is calculated from the momentum theorem, and energy loss rate due to vortex shedding in the wake is calculated from the principle of conservation of mechanical energy. Total power required to maintain the oscillations and hydrodynamic efficiency are also determined. In two dimensions, the results are obtained in closed form. In three dimensions, the distribution of vorticity on the lifting surface is also required as input to the calculations. Therefore, unsteady lifting-surface theory must be used as well. The analysis is applicable to oscillating lifting surfaces of arbitrary planform, aspect ratio and reduced frequency and does not require calculation of the leading-edge thrust.

1. Introduction

Certain types of animal locomotion in nature routinely achieve high propulsive efficiences. Those of particular interest to the aerodynamicist are the flapping flight of birds and lunate-tail swimming propulsion of many fast-moving fish. For high-Reynolds-number attached flows, the aerodynamics of such an oscillating lifting surface can be calculated from linearized inviscid unsteady aerodynamic theory (for small to moderate oscillation amplitudes). Prediction of the propulsive performance of such an oscillating lifting surface requires calculation of the energetic quantities, i.e. thrust, power required to maintain the oscillations, energy loss rate due to vortex shedding in the wake (wake energy) and hydrodynamic efficiency.

In two-dimensions, the energetic quantities for a harmonically oscillating airfoil have been calculated by von Kármán & Burgers (1935), Garrick (1936), and Lighthill (1970) for a rigid airfoil; by Wu (1961) and Siekman (1962, 1963) for a flexible airfoil; by Wu (1971 *a*) for a flexible airfoil in variable forward speed motion; and by Chopra (1976) for a rigid airfoil in heaving motion of large amplitude with small-amplitude pitching motion about the local path (rigid wake).

In three dimensions, the energetic quantities have been calculated by Chopra (1974) for a rigid wing using superposition of sinusoidal lifting ribbons; by Chopra & Kambe (1977) and Lan (1979) for rigid wings using numerical unsteady lifting surface theory; and by Ahmadi & Widnall (1983) for spanwise flexible wings using unsteady lifting-line theory which yields closed form results for low reduced frequencies and high aspect

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ratios. All of these calculations involve direct calculation of the energetic quantities including calculation of the leading-edge thrust (suction force).

This paper presents an alternative approach to the calculation of the energetics of oscillating lifting surfaces, using integral conservation laws. The momentum theorem is employed to calculate total thrust and the principle of conservation of mechanical energy is used to calculate energy loss rate in the wake (wake energy). Input power and hydrodynamic efficiency are also determined. Wake energy was first calculated in this way by von Kármán & Burgers (1935) and Garrick (1936) for a harmonically oscillating rigid airfoil. The present analysis is carried out in two and three dimensions for general and harmonic transverse oscillations of a flexible lifting surface.

In steady flow, application of the integral conservation laws gives results for lift and drag that depend only on the span distribution of circulation. For unsteady flow considerably more detail about the flow is required to calculate lift and drag from these conservation laws.

2. Energetics of an oscillating airfoil

2.1. *Thrust*

Consider a thin, two-dimensional airfoil in small-amplitude transverse oscillation in a uniform stream of inviscid incompressible fluid. We calculate thrust by applying the momentum theorem to the fluid contained within a fixed control volume V which is bounded on the inside by the airfoil surface σ and the wake surface S_w and on the outside by a far boundary S consisting of S_1, S_2, S_3 and S_4 (figure 1). The coordinate system (x, z) is fixed to the mean position of the airfoil and is stationary with respect to the control volume V. The airfoil semi-chord is denoted by c and the abscissa of the downstream end of the wake is L. When the wake extends beyond S_3, L is taken as the abscissa of S_3. The free-stream velocity U is in the positive x-direction.

With body forces neglected, the momentum theorem states that the force exerted by the fluid on the airfoil per unit span is given by

$$\boldsymbol{F}_B(t) = -\int_S p\boldsymbol{n}\,\mathrm{d}S - \int_{S+S_w+\sigma} \rho\boldsymbol{Q}(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_V \frac{\partial}{\partial t}(\rho\boldsymbol{Q})\,\mathrm{d}V, \tag{1}$$

where $\boldsymbol{Q}$ is the velocity vector and $\boldsymbol{n}$ is the unit normal vector at the boundaries pointing away from V. We consider the unsteady motion of the flexible mid-camber line of the airfoil (σ). To obtain the thrust all quadratic terms will be retained in the analysis and the actual non-planar airfoil and wake geometry must be considered to this order.

We substitute the perturbation velocity

$$\boldsymbol{q} = \boldsymbol{Q} - U\boldsymbol{i} = u\boldsymbol{i} + w\boldsymbol{k} \tag{2}$$

and the (incompressible) continuity equation

$$\int_{S+S_w+\sigma} (\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S = 0 \tag{3}$$

into (1) to obtain

$$\boldsymbol{F}_B(t) = -\int_S (p-p_\infty)\boldsymbol{n}\,\mathrm{d}S - \rho\int_{S+S_w+\sigma} \boldsymbol{q}(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S - \rho\int_V \frac{\partial}{\partial t}\boldsymbol{q}\,\mathrm{d}V, \tag{4}$$

where u and w are the perturbation velocity components in the x and z directions

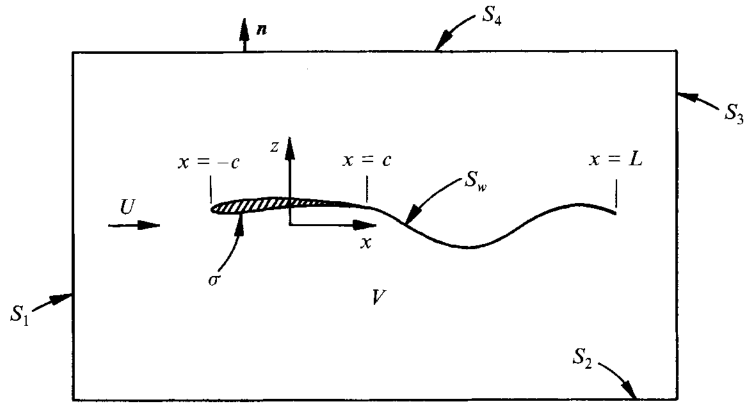

FIGURE 1. Control volume for the momentum theorem in two dimensions.

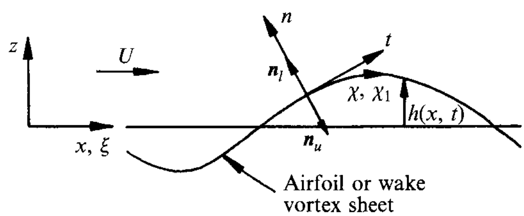

FIGURE 2. Schematic of airfoil and wake geometry.

respectively and $\boldsymbol{i}$ and $\boldsymbol{k}$ are the corresponding unit vectors. Thrust per unit span is the negative x-component of this vector force, i.e.

$$T(t) = -\int_{S_1}(p-p_\infty)\,\mathrm{d}S + \int_{S_3}(p-p_\infty)\,\mathrm{d}S + \rho\int_{S+S_w+\sigma} u(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S + \rho\int_V \frac{\partial}{\partial t}u\,\mathrm{d}V. \quad (5)$$

In the first two integrals we use the Bernoulli equation:

$$p-p_\infty = -\rho\left[\frac{\partial\phi}{\partial t} + Uu + \tfrac{1}{2}(u^2+w^2)\right], \quad (6)$$

where ϕ is the perturbation velocity potential ($\boldsymbol{q} = \nabla\phi$). We convert the volume integral in (5) to a surface integral by use of the gradient theorem.

$$\int_V \frac{\partial u}{\partial t}\mathrm{d}V = -\int_{S_1}\frac{\partial\phi}{\partial t}\mathrm{d}S + \int_{S_3}\frac{\partial\phi}{\partial t}\mathrm{d}S + \int_{S_w+\sigma}\frac{\partial}{\partial t}(\Delta\phi)\,\boldsymbol{i}\cdot\boldsymbol{n}_u\,\mathrm{d}S + \int_{LE}\frac{\partial\phi}{\partial t}\boldsymbol{i}\cdot\boldsymbol{n}\,\mathrm{d}S, \quad (7)$$

where $\Delta\phi = \phi_u - \phi_l$ is the jump in velocity potential across the airfoil or wake and LE denotes the leading edge. $(\,)_u$ and $(\,)_l$ denote, respectively, the upper and lower airfoil or wake surfaces. Wu (1961, 1971*a*) has shown that near the leading edge of an oscillating airfoil ϕ and $\partial\phi/\partial t$ remain bounded. Hence, the integral around the leading edge which is over a vanishingly small region is identically zero. A similar integral around the trailing edge of the wake is also zero.

The integral over $S_w+\sigma$ in (7) is to be carried out only on the upper side of these surfaces. Figure 2 depicts a segment of the airfoil or wake vortex sheet where χ is the distance along the sheet and $h(x,t)$ is the lateral displacement of the sheet from the x-axis. It follows from the definition of the velocity potential that

$$\Delta\phi = \int_{LE}^{\chi}\gamma(\chi_1,t)\,\mathrm{d}\chi_1 = \int_{-c}^{x}\gamma(\xi,t)\,\mathrm{d}\xi + O(\epsilon^2), \quad (8)$$

where γ is the vorticity per unit length and ϵ is a small non-dimensional parameter denoting the order of the perturbations. The second form is to be carried out along the linearized vortex sheet which lies on the x-axis. The unit normal vector at the non-planar sheet is given by

$$\boldsymbol{n}_u = \frac{\partial h}{\partial x}\boldsymbol{i} - \boldsymbol{k} + O(\epsilon^2),$$
$$\boldsymbol{n}_l = -\boldsymbol{n}_u. \tag{9}$$

Substituting (8) and (9) into the third integral in (7) and integrating by parts, we obtain

$$\int_{S_w+\sigma} \frac{\partial}{\partial t}(\Delta\phi)\,\boldsymbol{i}\cdot\boldsymbol{n}_u\,\mathrm{d}S = h_w(L,t)\frac{\partial}{\partial t}\Gamma(L,t) - \int_{-c}^{L}\frac{\partial}{\partial t}[\gamma(x,t)]\,h(x,t)\,\mathrm{d}x, \tag{10}$$

where $(\;)_w$ denotes the wake and

$$\Gamma(L,t) = \int_{-c}^{L}\gamma(\xi,t)\,\mathrm{d}\xi. \tag{11}$$

The volume integral in (5) then becomes

$$\int_V \frac{\partial u}{\partial t}\mathrm{d}V = -\int_{S_1}\frac{\partial\phi}{\partial t}\mathrm{d}S + \int_{S_3}\frac{\partial\phi}{\partial t}\mathrm{d}S + h_w(L,t)\frac{\partial}{\partial t}\Gamma(L,t) - \int_{-c}^{L}\frac{\partial}{\partial t}[\gamma(x,t)]\,h(x,t)\,\mathrm{d}x. \tag{12}$$

Next, we consider the momentum flux integral over $S+S_w+\sigma$ in (5). In terms of the perturbation velocities, the part of the integral which is over S becomes

$$\int_S u(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S = -\int_{S_1}(Uu+u^2)\,\mathrm{d}S - \int_{S_2} uw\,\mathrm{d}S + \int_{S_3}(Uu+u^2)\,\mathrm{d}S + \int_{S_4} uw\,\mathrm{d}S. \tag{13}$$

It follows from (2), (9) and the downwash at σ and S_w, namely

$$\left[\frac{\partial}{\partial t} + (U+u)\frac{\partial}{\partial x}\right]h(x,t) = w(x,t) \quad (z = h(x,t)) \tag{14}$$

that, on these surfaces,

$$(\boldsymbol{Q}\cdot\boldsymbol{n})_u = -(\boldsymbol{Q}\cdot\boldsymbol{n})_l = -\frac{\partial h}{\partial t} + O(\epsilon^2). \tag{15}$$

Using this result, the momentum flux integral over σ and S_w becomes

$$\int_{S_w+\sigma} u(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S = -\int_{S_w+\sigma}\Delta u\frac{\partial h}{\partial t}\mathrm{d}S + \int_{LE} u(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S, \tag{16}$$

where $\Delta u = u_u - u_l$. The integral around the leading edge, being over a vanishingly small region, is identically zero since at the leading edge $(\boldsymbol{Q}\cdot\boldsymbol{n})$ is finite and u has an integrable singularity. A similar integral around the trailing edge of the wake is likewise zero. It can be shown using the intrinsic coordinate system (s,n) (figure 2), that

$$\Delta u = \gamma + O(\epsilon^3). \tag{17}$$

Hence,

$$\int_{S_w+\sigma} u(\boldsymbol{Q}\cdot\boldsymbol{n})\,\mathrm{d}S = -\int_{-c}^{L}\gamma(x,t)\frac{\partial}{\partial t}h(x,t)\,\mathrm{d}x. \tag{18}$$

Substituting the above results into (5), we obtain

$$T(t) = \tfrac{1}{2}\rho \int_{S_1} (w^2-u^2)\,\mathrm{d}S - \tfrac{1}{2}\rho \int_{S_3} (w^2-u^2)\,\mathrm{d}S - \rho \int_{S_2} uw\,\mathrm{d}S + \rho \int_{S_4} uw\,\mathrm{d}S$$
$$+\rho h_w(L,t)\frac{\partial}{\partial t}\Gamma(L,t) - \rho \int_{-c}^{L} \frac{\partial}{\partial t}[\gamma(x,t)\,h(x,t)]\,\mathrm{d}x. \quad (19)$$

It can be shown that, as the far boundary S is removed to infinity, the integrals over S_1, S_2 and S_4 in (19) vanish. Hence,

$$T(t) = -\tfrac{1}{2}\rho \int_{S_3} (w^2-u^2)\,\mathrm{d}S + \rho h_w(L,t)\frac{\partial}{\partial t}\Gamma(L,t) - \rho \int_{-c}^{L} \frac{\partial}{\partial t}[\gamma(x,t)\,h(x,t)]\,\mathrm{d}x. \quad (20)$$

This result is valid for arbitrary small-amplitude transverse oscillations of the airfoil and vanishes in the limit of steady flow in accordance with d'Alembert's paradox.

For harmonic oscillations, we use the unsteady airfoil theory of Schwarz (1940; Schwarz's theory is also presented in Bisplinghoff, Ashley & Halfman 1955) together with (20) to calculate thrust. Here, in analogy with steady flow, we refer to S_3 as the Trefftz plane ($L \rightarrow \infty$). The amplitude of the airfoil circulation and wake vorticity are, respectively, given by

$$\tilde{\Gamma} = c\,\mathrm{e}^{-jk}\tilde{\Omega}, \quad (21)$$

$$\tilde{\gamma}_w(x) = -jk\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}x}, \quad (22)$$

where $k = \omega c/U$ is the reduced frequency, $\tilde{\Omega}$ is the reduced circulation, j is the temporal complex unit $\sqrt{-1}$, and $(\tilde{\ })$ denotes complex amplitude with respect to j.

$$\tilde{\Omega}(k) = \frac{4\displaystyle\int_{-c}^{c}\left[\frac{c+\xi}{c-\xi}\right]^{\frac{1}{2}} \tilde{W}_0(\xi)\,\mathrm{d}\xi}{\pi jck[H_1^{(2)}(k)+jH_0^{(2)}(k)]}, \quad (23)$$

where $H_n^{(2)}$ is the Hankel function of the second kind of order n. $\tilde{W}_0$ is the prescribed linearized downwash at the airfoil:

$$W_0(x,t) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\tilde{h}_a(x)\,\mathrm{e}^{j\omega t} \quad (|x| \leqslant c, \quad z = o\pm), \quad (24)$$

where $(\ )_a$ denotes the airfoil.

It follows from (11), (21) and (22) that

$$\tilde{\Gamma}(L) = c\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}L}, \quad (25)$$

$$\frac{\partial}{\partial t}\Gamma(L,t) = -U\gamma_w(L,t). \quad (26)$$

Substituting (26) into (20), we obtain

$$T(t) = -\tfrac{1}{2}\rho \int_{S_3} (w^2-u^2)\,\mathrm{d}S - \rho U h_w(L,t)\,\gamma_w(L,t) - \rho \int_{-c}^{L} \frac{\partial}{\partial t}[\gamma(x,t)\,h(x,t)]\,\mathrm{d}x. \quad (27)$$

The average thrust per unit span is given by

$$\bar{T} = -\rho U\overline{h_w(L,t)\,\gamma_w(L,t)} - \tfrac{1}{2}\rho \int_{S_3} (\overline{w^2} - \overline{u^2})\,\mathrm{d}S. \quad (28)$$

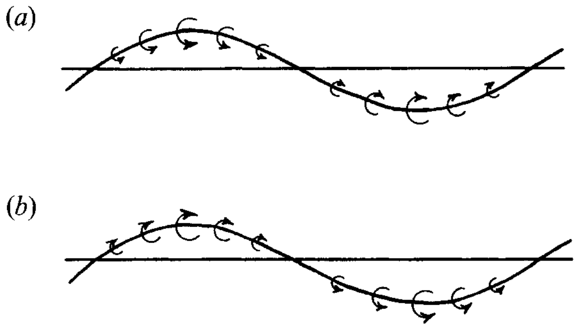

FIGURE 3. (*a*) Thrust-type wake ($\alpha-\beta=\pi$); (*b*) drag-type wake ($\alpha-\beta=0$), (the strength and sense of local vorticity is indicated by curved arrows).

It is shown in Appendix B that for harmonic oscillation in the far wake

$$\overline{u^2}=\overline{w^2}. \tag{29}$$

Hence, the average thrust per unit span reduces to

$$\bar{T}=-\rho U\overline{h_w(L,t)\,\gamma_w(L,t)}. \tag{30}$$

In the absence of the mechanism of diffusion, the wake vorticity is convected downstream without change. Hence, $\gamma_w(L,t)$ is given by (22). The determination of $h_w(L,t)$, on the other hand, requires some calculation, since the ultimate displacement of a wake element is determined by its entire past history, determined by the varying field of downwash along its path from the trailing edge to the Trefftz plane; $h_w(L,t)$ of the wake is calculated in the next section.

We end this section with an examination of the phase relationship between far wake displacement and vorticity. Let,

$$h_w(L,t)=|h_w|\,e^{j\alpha}\,e^{j\omega t}, \tag{31}$$

$$\gamma_w(L,t)=|\gamma_w|\,e^{j\beta}\,e^{j\omega t}, \tag{32}$$

where the amplitude and phase of $h_w(L,t)$ and $\gamma_w(L,t)$ are, respectively, denoted by $|h_w|$ and $|\gamma_w|$, and α and β. Substituting these in (30), we obtain

$$\bar{T}=-\tfrac{1}{2}\rho U|h_w|\,|\gamma_w|\cos(\alpha-\beta). \tag{33}$$

For the purpose of discussion, we assume that $|h_w|$ and $|\gamma_w|$ are fixed and consider the following special cases.

(i) If $\alpha-\beta=\frac{1}{2}\pi$, $\bar{T}=0$. It can be seen from the results of Appendix B that the self-induced downwash of a linearized wake with sinusoidally varying strength is out of phase with the vorticity distribution by $\frac{1}{2}\pi$. Hence, the self-induced displacement of such a wake is also out of phase with the vorticity by $\frac{1}{2}\pi$ and the corresponding contribution to the average thrust is zero.

(ii) If $\alpha-\beta=\pi$, we have the case of maximum average thrust (for fixed $|h_w|$ and $|\gamma_w|$). This does not necessarily correspond to the optimum motion which is the solution of a constrained variational problem (Ahmadi & Widnall 1983; Wu 1971*b*).

(iii) If $\alpha-\beta=0$, we have the case of maximum average drag (for fixed $|h_w|$ and $|\gamma_w|$).

(iv) Cases with $0<\alpha-\beta<\frac{1}{2}\pi$ correspond to those shapes and motions of the airfoil which, in the average, produce drag, whereas cases with $\frac{1}{2}\pi<\alpha-\beta<\pi$ correspond to thrust-producing configurations. Figure 3 shows a thrust- and a drag-type far wake corresponding to cases (ii) and (iii) in the above.

The equation of average thrust can be interpreted as the average flux of momentum associated with the wake vortices crossing the Trefftz plane. For positive thrust the displacement of wake vorticity gives rise to a net flux of momentum in the downstream direction; for drag to a net flux of momentum in the upstream direction. The Kármán vortex street is a drag-type wake.

2.2. *Asymptotic wake displacement*

We showed in the above that the average thrust per unit span of a harmonically oscillating airfoil is proportional to the time average of the far-wake displacement and vorticity. Since thrust is $O(\epsilon^2)$ and the wake vorticity is $O(\epsilon)$, we need to determine the wake displacement only to $O(\epsilon)$. This can be accomplished using a linearized (planar) wake model as shown below.

The linearized downwash at the wake is given by

$$W_w(x,t) = \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) h_w(x,t) \quad (x \geqslant c, z = 0\pm). \tag{34}$$

For harmonic motion this reduces to

$$\tilde{W}_w(x) = j\omega \tilde{h}_w(x) + U\frac{\mathrm{d}}{\mathrm{d}x}\tilde{h}_w(x) \quad (x \geqslant c, z = 0\pm). \tag{35}$$

To obtain the wake displacement, we invert (35) by multiplying through by $\mathrm{e}^{j\bar{\omega}x}$ and integrating from the trailing edge up to $x > c$.

$$\tilde{h}_w(x) = \tilde{h}(c)\,\mathrm{e}^{-j\bar{\omega}(x-c)} + U^{-1}\int_c^x \tilde{W}_w(\xi)\,\mathrm{e}^{-j\bar{\omega}(x-\xi)}\,\mathrm{d}\xi \quad (x \geqslant c). \tag{36}$$

Downwash at the plane of the wake, $\tilde{W}_w$, is given by

$$\tilde{W}_w(\xi) = \frac{-1}{2\pi}\int_{-c}^{c}\frac{\tilde{\gamma}_a(x)}{\xi - x}\mathrm{d}x + \frac{jk}{2\pi}\tilde{\Omega}\oint_c^\infty \frac{\mathrm{e}^{-j\bar{\omega}x}}{\xi - x}\mathrm{d}x \quad (\xi \geqslant c, z = 0\pm), \tag{37}$$

where the first term is the contribution of the airfoil and the second is that of the wake. The vorticity distribution on the airfoil is given by (Schwarz 1940)

$$\tilde{\gamma}_a(x) = \frac{2}{\pi}\left[\frac{c-x}{c+x}\right]^{\frac{1}{2}}\left\{\oint_{-c}^{c}\left[\frac{c+\lambda}{c-\lambda}\right]^{\frac{1}{2}}\frac{\tilde{W}_0(\lambda)}{x-\lambda}\mathrm{d}\lambda + \tfrac{1}{2}jk\tilde{\Omega}\int_c^\infty\left[\frac{\lambda+c}{\lambda-c}\right]^{\frac{1}{2}}\frac{\mathrm{e}^{-j\omega\lambda}}{x-\lambda}\mathrm{d}\lambda\right\} \quad (|x| \leqslant c). \tag{38}$$

Substituting (38) into (37), interchanging the order of integrations in the first two terms and making use of the first two integrals in Appendix A, we obtain

$$\tilde{W}_w(\xi) = \frac{-1}{\pi}\left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}}\left\{\int_{-c}^{c}\left[\frac{c+\lambda}{c-\lambda}\right]\frac{\tilde{W}_0(\lambda)}{\lambda-\xi}\mathrm{d}\lambda + \tfrac{1}{2}jk\tilde{\Omega}\oint_c^\infty\left[\frac{\lambda+c}{\lambda-c}\right]^{\frac{1}{2}}\frac{\mathrm{e}^{-j\bar{\omega}\lambda}}{\lambda-\xi}\mathrm{d}\lambda\right\}$$
$$(\xi \geqslant c, z = 0\pm), \tag{39}$$

where the first term is the quasi-steady contribution and the second term represents all direct and indirect contributions from the wake.

Substituting (39) into (36), we obtain

$$\tilde{h}_w(x) = \mathrm{e}^{-j\bar{\omega}x}\left\{\tilde{h}(c)\,\mathrm{e}^{jk} - \frac{1}{\pi U}\int_c^x \mathrm{d}\xi\int_{-c}^{c}\mathrm{d}\lambda\left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}}\left[\frac{c+\lambda}{c-\lambda}\right]^{\frac{1}{2}}\frac{\tilde{W}_0(\lambda)\,\mathrm{e}^{j\bar{\omega}\xi}}{\lambda-\xi}\right.$$
$$\left. - \frac{jk\tilde{\Omega}}{2\pi U}\int_c^x \mathrm{d}\xi\oint_c^\infty \mathrm{d}\lambda\left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}}\left[\frac{\lambda+c}{\lambda-c}\right]\frac{\mathrm{e}^{j\bar{\omega}(\xi-\lambda)}}{\lambda-\xi}\right\} \quad (x \geqslant c). \tag{40}$$

Here, the first term is the displacement of a rigid wake which is the sinusoidal trace of the trailing edge. The second term is due to the quasi-steady effects and the third term represents all direct and indirect contributions from the wake.

The asymptotic displacement of the wake $\tilde{h}(L)$ is obtained from (40) by setting $x = \infty$ in the upper limit of the integrals.

$$\tilde{h}_w(L) = e^{-j\bar{\omega}x}\left\{\tilde{h}(c)\,e^{jk} - \frac{1}{\pi U}\int_c^\infty d\xi \int_{-c}^{c} d\lambda \left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}}\left[\frac{c+\lambda}{c-\lambda}\right]^{\frac{1}{2}}\frac{\tilde{W}_0(\lambda)\,e^{j\bar{\omega}\xi}}{\lambda-\xi}\right.$$
$$\left. + \frac{jk\tilde{\Omega}}{\pi U}\int_c^\infty d\xi \oint_c^\infty d\lambda \left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}}\left[\frac{\lambda+c}{\lambda-c}\right]^{\frac{1}{2}}\frac{e^{j\bar{\omega}(\xi-\lambda)}}{\xi-\lambda}\right\}. \quad (41)$$

The double integral in the last term is evaluated in Appendix A. In the limit of steady flow, this asymptotic displacement of the wake contains a logarithmic singularity which arises from the second term of (41) and can be expressed as $\log L$ or $\log k$.

Substituting (A 11), (41) and (22) into (30) and introducing the non-dimensional quantities

$$x^* = x/c, \quad h^* = h/c, \quad W_0^* = W_0/U, \quad \Omega^* = \Omega/U, \quad C_T = \bar{T}/\tfrac{1}{4}\pi\rho U^2 c], \quad (42)$$

we obtain the following expression for the thrust coefficient of a harmonically oscillating airfoil

$$C_T = \frac{-2}{\pi}kR_j\left\{j(\tilde{\Omega}^*)^*\tilde{h}^*(1)\,e^{jk} - \frac{j}{\pi}(\tilde{\Omega}^*)^*\int_1^\infty d\xi^* \int_{-1}^{1} d\lambda^* \left[\frac{\xi^*-1}{\xi^*+1}\right]^{\frac{1}{2}}\left[\frac{1+\lambda^*}{1-\lambda^*}\right]^{\frac{1}{2}}\frac{\tilde{W}_0^*(\lambda^*)\,e^{jk\xi^*}}{\lambda^*-\xi^*}\right.$$
$$\left. - \tfrac{1}{8}\pi k|\tilde{\Omega}^*|^2\,[J_0^2(k) + Y_0^2(k)]\right\}. \quad (43)$$

where R_j denotes real part with respect to j and $(\,)^*$ denotes complex conjugate. This form has the advantage that it relates the thrust to the airfoil shapes and motions.

We end this section with an example. Consider an airfoil in heaving motion where

$$h_a(x,t) = h_0\,e^{j\omega t} \quad (|x| \leqslant c), \quad (44)$$

$$\tilde{W}_0^* = jkh_0^* \quad (|x^*| \leqslant 1, z^* = 0\pm). \quad (45)$$

Substituting (45) into (23) and (43) and using the following integrals (these can be obtained from certain integrals on pp. 251–252 of Ashley & Landahl 1965)

$$\int_{-1}^{1}\left[\frac{1+\lambda^*}{1-\lambda^*}\right]^{\frac{1}{2}}\frac{d\lambda^*}{\lambda^*-\xi^*} = \pi\left\{1-\left[\frac{\xi^*+1}{\xi^*-1}\right]^{\frac{1}{2}}\right\} \quad (\xi^* \geqslant 1), \quad (46)$$

$$\int_1^\infty \left\{\left[\frac{\xi^*-1}{\xi^*+1}\right]^{\frac{1}{2}}-1\right\}e^{jk\xi^*}\,d\xi^* = \{e^{jk}/(jk) - \tfrac{1}{2}\pi[H_1^{(2)*}(k) + jH_0^{(2)*}(k)]\}, \quad (47)$$

we obtain the known result for the thrust coefficient of a heaving airfoil (see, e.g. Wu 1971*b*).

$$C_{T_H} = 4k^2 D(k)\,h_0^{*2}, \quad (48)$$

where $(\,)_H$ denotes heaving motion and

$$D(k) = [F^2(k) + G^2(k)], \quad (49)$$

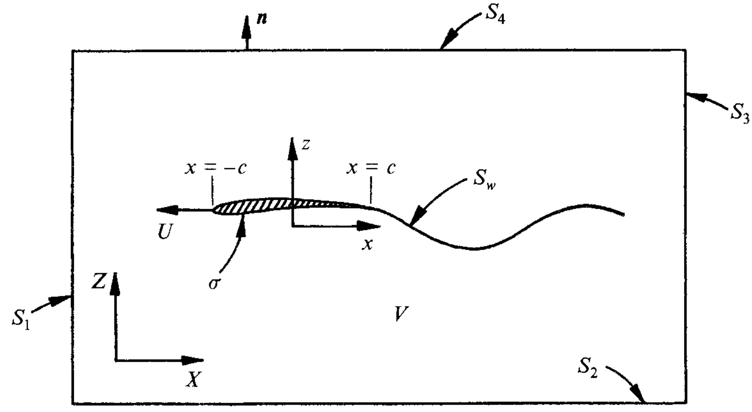

FIGURE 4. Control volume for conservation of energy in two dimensions.

F and G are, respectively, the real and imaginary parts of Theodorsen's function (Theodorsen 1935):

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k)+jH_0^{(2)}(k)}$$

$$= \frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2} - j\frac{(Y_1 Y_0+J_1 J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}. \tag{50}$$

Next, we calculate the wake energy.

2.3 *Wake energy*

We now consider the energy in the wake of the airfoil. Since the fluid is non-dissipative and incompressible, the work done by the airfoil on the fluid ultimately shows up in the far wake in the form of kinetic energy. We apply the principle of conservation of mechanical energy to the fluid in a fixed control volume V bounded on the inside by the airfoil and wake surfaces, σ and S_w, and on the outside by a far boundary S (consisting of S_1, S_2, S_3 and S_4) which is located infinitely far from the airfoil and wake, as shown in figure 4. The fluid contained in V is thus free of discontinuities.

The coordinate system (X, Z) is at rest with respect to the undisturbed fluid; the parallel coordinate system (x, z) moves with the airfoil mean velocity U in the negative X-direction. The (x, z) observer measures a velocity field $\boldsymbol{Q}$ consisting of a free stream $U\boldsymbol{i}$ and a perturbation field $\boldsymbol{q}$. The control volume V is at rest with respect to the (X, Z) frame.

With body forces neglected, the balance of energy for the fluid in V with respect to the (X, Z) frame is

$$\int_V \frac{\partial}{\partial t}(\tfrac{1}{2}\rho|\boldsymbol{q}|^2)\,\mathrm{d}V = -\int_{S+S_w+\sigma} p\boldsymbol{n}\cdot\boldsymbol{q}\,\mathrm{d}S, \tag{51}$$

which states that the rate of change of the total kinetic energy of the fluid in V is equal to the rate of work of the external forces on the same fluid. Since pressure is continuous across the wake, the integral over S_w is zero. Also it can be shown that, for S infinitely removed from the airfoil and the wake, the integral over S vanishes as well. The integral over the airfoil surface can be written as the sum of integrals over the upper and lower surfaces and the leading and trailing edges of the airfoil, namely

$$-\int_\sigma p\boldsymbol{n}\cdot\boldsymbol{q}\,\mathrm{d}S = -\left[\int_{\sigma_u} + \int_{\sigma_l} + \int_{LE} + \int_{TE}\right] p\boldsymbol{n}\cdot\boldsymbol{q}\,\mathrm{d}S, \tag{52}$$

where TE denotes the trailing edge of the airfoil. The latter integral is identically zero owing to the Kutta condition. The integral around the leading edge is the rate of work of the leading-edge suction force T_s on the fluid, i.e. UT_s.

In (52), $\boldsymbol{q}$ is the velocity of the airfoil mid-camber line

$$\boldsymbol{q} = -U\boldsymbol{i} + \frac{\partial}{\partial t} h_a \boldsymbol{k}, \tag{53}$$

and $\boldsymbol{n}$ is the unit normal vector at the airfoil, with respect to the (X, Z) frame:

$$\boldsymbol{n}_u = \frac{\partial}{\partial X} h_a \boldsymbol{i} - \boldsymbol{k} + O(\epsilon^2),$$

$$\boldsymbol{n}_l = -\boldsymbol{n}_u. \tag{54}$$

Using (53) and (54), the integrals over the upper and lower surfaces of the airfoil may be combined to obtain

$$-\left[\int_{\sigma_u} + \int_{\sigma_l}\right] p\boldsymbol{n}\cdot\boldsymbol{q}\,\mathrm{d}S = -UT_p - \int_{LE}^{TE} \Delta p \frac{\partial}{\partial t} h_a\,\mathrm{d}X, \tag{55}$$

where
$$T_p = \int_{LE}^{TE} \Delta p \frac{\partial}{\partial X} h_a\,\mathrm{d}X \tag{56}$$

is the thrust contribution from the normal force at the airfoil.

Combining the above results and noting that $T = T_s + T_p$, (51) becomes

$$\int_V \frac{\partial}{\partial t}(\tfrac{1}{2}\rho|\boldsymbol{q}|^2)\,\mathrm{d}V = -UT - \int_{LE}^{TE} \Delta p \frac{\partial}{\partial t} h_a\,\mathrm{d}X. \tag{57}$$

Averaging this over the time interval τ, we obtain

$$\frac{1}{\tau}\Delta(KE) = -U\bar{T} - \overline{\int_{LE}^{TE} \Delta p \frac{\partial}{\partial t} h_a\,\mathrm{d}X}, \tag{58}$$

where
$$\frac{1}{\tau}\Delta(KE) = \frac{1}{\tau}\left\{\left[\int_V \tfrac{1}{2}\rho|\boldsymbol{q}|^2\,\mathrm{d}V\right]_{t=t_0+\tau} - \left[\int_V \tfrac{1}{2}\rho|\boldsymbol{q}|^2\,\mathrm{d}V\right]_{t=t_0}\right\} \tag{59}$$

is the average rate of change of the total kinetic energy of the fluid in V during τ. t_0 is an arbitrary constant and $(\overline{\ \ })$ denotes time average.

Equation (58) states that the average power required to maintain the airfoil oscillations is equal to the average rate of work of thrust plus the average rate of increase of the kinetic energy of the fluid. The latter is the average energy loss rate, i.e. energy imparted to the fluid which cannot be recovered. Denoting the latter by $\bar{E}$ and the average power required by $\bar{P}$, (57) becomes

$$\bar{P} = U\bar{T} + \bar{E} \quad \text{or} \quad C_P = C_T + C_E. \tag{60}$$

where C_E and C_P are, respectively, the energy loss rate and input power coefficients which are defined by

$$C_E = \bar{E}/[\tfrac{1}{4}\pi\rho U^3 c], \quad C_P = \bar{P}/[\tfrac{1}{4}\pi\rho U^3 c]. \tag{61}$$

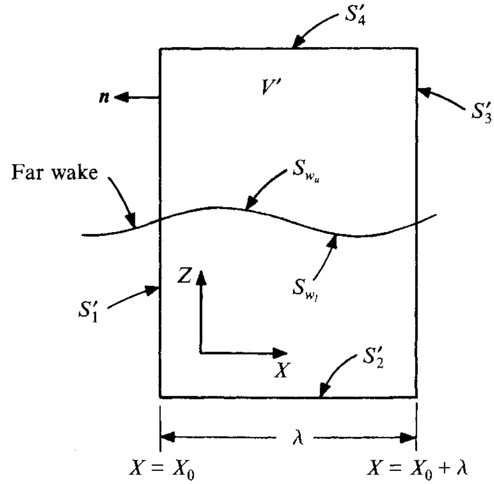

FIGURE 5. Control volume for calculation of wake energy.

Thus, the input power is partly used to produce thrust, and thereby useful work, and partly wasted in generating a wake of vorticity. The hydrodynamic efficiency of the motion is defined as the ratio of the useful power to the input power, i.e.

$$\eta = C_T/C_P = 1 - C_E/C_P. \tag{62}$$

For harmonic oscillations of radian frequency ω, we choose τ to be the period $2\pi/\omega$, during which time one wavelength $\lambda = 2\pi U/\omega$ of the periodic wake is generated. $\Delta(KE)$ is then the kinetic energy content of one wavelength of the far wake. This can be seen by comparing the flow field at times t and $t+\tau$. The only difference is that the far wake for $t+\tau$ is one wavelength longer than that for t. Hence, $\Delta(KE)$ is the kinetic energy content of a slice of the far wake of length λ, as shown in figure 5. From potential flow theory, the kinetic energy of the fluid in this volume is given by

$$\Delta(KE) = \frac{1}{2}\rho \int_{S'} \phi \frac{\partial}{\partial n}\phi \, \mathrm{d}S, \tag{63}$$

where S' consists of S_1', S_2', S_3', S_4', S_{w_u}, and S_{w_l}. Owing to the periodicity of flow properties in the far wake (Appendix B), the integrals over S_1' and S_3' cancel each other out. As the lateral boundaries S_2' and S_4' are removed to infinity, the integrals over these surfaces vanish since from Appendix B

$$\phi, u, w \sim \mathrm{e}^{-\bar{\omega}|z|}. \tag{64}$$

The integrals over the upper and lower wake surfaces can be combined using (54) to obtain

$$\Delta(KE) = -\tfrac{1}{2}\rho \int_{X_0}^{X_0+\lambda} \Delta\phi \left.\frac{\partial \phi}{\partial Z}\right|_{Z=0} \mathrm{d}X, \tag{65}$$

where we have neglected terms of $O(\epsilon^3)$ and X_0 is an arbitrary constant. The waviness of the wake, thus, does not appear to this order. We note that, while a planar wake is adequate for calculating the wake energy, the actual wake geometry must be considered for calculating thrust from the momentum theorem. This is because energy is a scalar quantity, whereas momentum is a vector quantity which is sensitive to changes in direction.

Equation (65) is essentially the spatial average with respect to X over the interval λ (analogous to the time average) of $\Delta\phi$ and $\partial\phi/\partial Z$. Denoting this spatial average by $(\bar{\ })$, we have

$$\Delta(KE) = -\tfrac{1}{2}\rho\lambda\,\overline{\overline{\Delta\phi\,\partial\phi/\partial Z}}|_{Z=0}. \tag{66}$$

For convenience, we evaluate $\Delta\phi$ and $\partial\phi/\partial Z$ with respect to the moving frame and then transform the results to the stationary frame. With respect to the (x, z) frame, it follows from (8), (21) and (22), that

$$\Delta\phi(x,t) = c\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}x}\,\mathrm{e}^{j\omega t} \quad (x \leqslant c). \tag{67}$$

The self-induced downwash at the plane of the wake is calculated in Appendix B:

$$\frac{\partial}{\partial Z}\phi(x,0,t) = -\tfrac{1}{2}k\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}x}\,\mathrm{e}^{j\omega t}. \tag{68}$$

The above results are transformed to the stationary frame using

$$x = X + Ut, \quad z = Z. \tag{69}$$

Thus,

$$\Delta\phi(X+Ut) = c\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}X}, \tag{70}$$

$$\left.\frac{\partial}{\partial Z}\phi(X+Ut)\right|_{Z=0} = -\tfrac{1}{2}k\tilde{\Omega}\,\mathrm{e}^{-j\bar{\omega}X} \tag{71}$$

Hence, the (X, Z) observer sees a steady flow field in the far wake.

It follows from (70), (71), (66) and (58) that the average energy loss rate is given by

$$\bar{E} = \tfrac{1}{8}\rho Uck|\tilde{\Omega}|^2. \tag{72}$$

Since Δp and $\partial h/\partial t$ are physically the same in both frames, the average power required can be expressed in the moving frame as

$$\bar{P} = -\overline{\int_{-c}^{c} \Delta p(x,t)\frac{\partial}{\partial t}h_a(x,t)\,\mathrm{d}x}. \tag{73}$$

Introducing the non-dimensional quantities in (42) and

$$z^* = z/c, \quad t^* = t/(c/U), \quad \Delta C_p = \Delta p/(\tfrac{1}{2}\rho U^2), \tag{74}$$

(72) and (73) become

$$C_E = \frac{1}{2\pi}k|\tilde{\Omega}^*|^2, \tag{75}$$

$$C_P = \frac{-2}{\pi}\overline{\int_{-1}^{1}\Delta C_p(x^*,t^*)\frac{\partial}{\partial t^*}h_a^*(x^*,t^*)\,\mathrm{d}x^*}. \tag{76}$$

C_T is then calculated from (60).

Calculation of C_P requires the unsteady pressure distribution on the airfoil. Ashley & Landahl (1965) give an efficient method of calculating ΔC_p. Their equation (13-54) in the present notation is (a misprint has been corrected)

$$\Delta\tilde{C}_p(x^*) = \frac{4}{\pi}\left\{\left[\frac{1-x^*}{1+x^*}\right]^{\frac{1}{2}}\oint_{-1}^{1}\left[\frac{1+\xi^*}{1-\xi^*}\right]^{\frac{1}{2}}\frac{\tilde{W}_0^*(\xi^*)}{x^*-\xi^*}\mathrm{d}\xi^* + (1-x^{*2})^{\frac{1}{2}}\oint_{-1}^{1}\frac{jk\tilde{f}(\xi^*)}{(x^*-\xi^*)(1-\xi^{*2})^{\frac{1}{2}}}\mathrm{d}\xi^*\right.$$
$$\left. + [1-C(k)]\left[\frac{1-x^*}{1+x^*}\right]^{\frac{1}{2}}\int_{-1}^{1}\left[\frac{1+\xi^*}{1-\xi^*}\right]^{\frac{1}{2}}\tilde{W}_0^*(\xi^*)\,\mathrm{d}\xi^*\right\}, \tag{77}$$

where $\tilde{f}$ is the auxiliary function

$$\tilde{f}(x^*) = \int_{-1}^{x^*} \tilde{W}_0^*(\xi_1^*)\, d\xi_1^*. \tag{78}$$

We end this section with two examples. First, we consider an airfoil in heave where (see (44) and (45))

$$\tilde{W}_0^* = jkh_0^* \quad (|x^*| \leqslant 1, z^* = 0\pm).$$

Using the identity

$$B(k) \equiv F-(F^2+G^2) = \frac{2}{\pi k[(J_1+Y_0)^2+(Y_1-J_0)^2]} \tag{79}$$

from Garrick (1936), the energy loss rate is obtained from (75),

$$C_{E_H} = 4k^2 B(k)\, h_0^{*2}. \tag{80}$$

The pressure distribution on the airfoil is obtained from (77),

$$\Delta\tilde{C}_p(x^*) = -4jkh_0^*\left\{C(k)\left[\frac{1-x^*}{1+x^*}\right]^{\frac{1}{2}} + jk(1-x^{*2})^{\frac{1}{2}}\right\}, \tag{81}$$

where we have made use of certain integrals from Van Dyke (1956). The input power is obtained from (76),

$$C_{P_H} = 4k^2 F(k)\, h_0^{*2}. \tag{82}$$

The thrust is calculated from (60) and found to be the same as in (48), as expected. The hydrodynamic efficiency for the motion is given by

$$\eta_H = D(k)/F(k). \tag{83}$$

Secondly, we consider an airfoil in pitching motion about the midchord where

$$h_a(x,t) = \alpha x\, e^{j\omega t} \quad (|x| \leqslant c), \tag{84}$$

$$\tilde{W}_0^*(x^*) = jk\alpha x^* + \alpha \quad (|x^*| \leqslant 1, z^* = 0\pm). \tag{85}$$

The energy loss rate is obtained from (75),

$$C_{E_P} = (4+k^2)\, B(k)\, \alpha^2. \tag{86}$$

From (77),

$$\Delta\tilde{C}_p(x^*) = -4\alpha\left\{[\tfrac{1}{2}jk + (1+\tfrac{1}{2}jk)\, C(k) + jkx^*]\left[\frac{1-x^*}{1+x^*}\right]^{\frac{1}{2}} + jk(1+\tfrac{1}{2}jkx^*)(1-x^{*2})^{\frac{1}{2}}\right\}, \tag{87}$$

where we have made use of certain integrals from Van Dyke (1956). The input power is then obtained from (76).

$$C_{P_P} = k[k(1-F)-2G]\,\alpha^2, \tag{88}$$

where $(\,)_P$ denotes pitching motion. The thrust is obtained from (60),

$$C_{T_P} = 4k^2\left[\left(\frac{1}{k^2}+\tfrac{1}{4}\right)D(k) - \left(\frac{1}{k^2}+\tfrac{1}{2}\right)F(k) - (\tfrac{1}{2}k)\, G(k) + \tfrac{1}{4}\right]\alpha^2. \tag{89}$$

The corresponding hydrodynamic efficiency η is obtained from (88) and (89).

The above results for the energetics of an airfoil in heave and pitch are in complete agreement with the known results obtained by direct calculation and calculation of

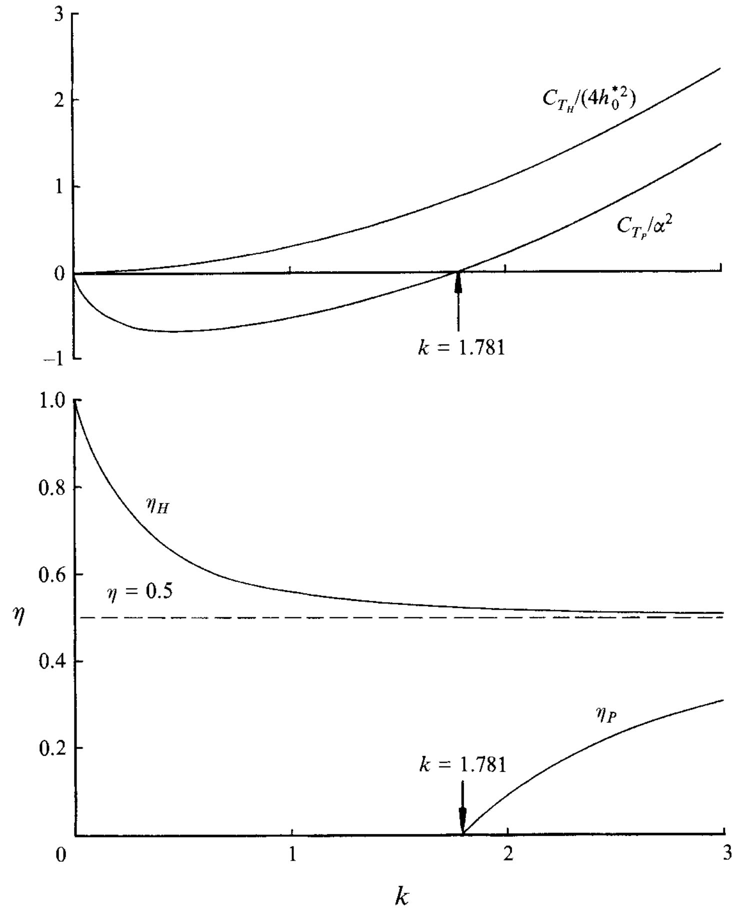

FIGURE 6. Thrust coefficient and hydrodynamic efficiency for an airfoil in pitch and heave.

leading-edge thrust (see, e.g. Wu 1971*b*). Thrust and hydrodynamic efficiency of heave and pitch are plotted in figure 6. Heaving motion produces thrust at all reduced frequencies. Pitching motion, on the other hand, produces drag except for $k > 1.781$. The efficiency of heaving motion starts at 100% for $k = 0$, drops off rapidly with increasing k and approaches 50% as $k \to \infty$. The efficiency of the pitching motion is defined only for $k > 1.781$, where it increases monotonically and approaches 50% as $k \to \infty$. Much higher efficiencies can be achieved from suitable combinations of pitch and heave (see Ahmadi & Widnall 1983; Wu 1971*b*).

3. Extension to three dimensions

We now extend the analysis to three dimensions to calculate the energetic quantities for flexible lifting surfaces of finite span. Again, we use the momentum theorem to calculate total thrust and conservation of energy to calculate wake energy.

3.1. *Thrust*

Consider a thin, almost-planar wing of finite span undergoing small-amplitude transverse oscillations in a uniform stream. To calculate the total thrust from the momentum theorem, (1), we select a far boundary S consisting of a right circular

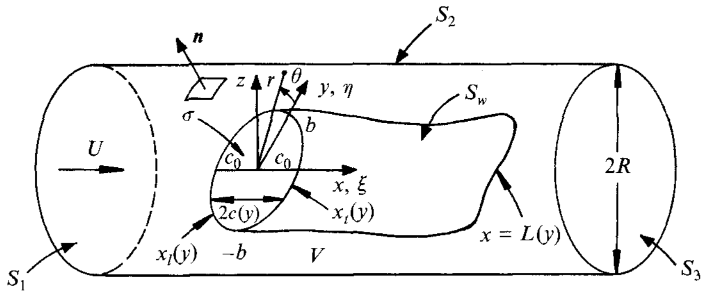

FIGURE 7. Control volume for the momentum theorem in three dimensions.

cylinder S_2 which is parallel to the main flow and two circular disks S_1 and S_3 of radius R, as shown in figure 7. We substitute the perturbation velocity

$$q = Q - Ui = ui + vj + wk, \tag{90}$$

and the continuity equation into (1), take the x-component of the result and use the Bernoulli equation and the gradient theorem to obtain the total thrust.

$$T(t) = \tfrac{1}{2}\rho \int_{S_1} (v^2 + w^2 - u^2)\, dS + \rho \int_{S_2} (uv\cos\theta + uw\sin\theta)\, dS$$
$$-\tfrac{1}{2}\rho \int_{S_3} (v^2 + w^2 - u^2)\, dS + \rho \int_{S_w+\sigma} \frac{\partial}{\partial t}(\Delta\phi)\, \boldsymbol{i}\cdot\boldsymbol{n}_u\, dS + \rho \int_{S_w+\sigma} \gamma(\boldsymbol{Q}\cdot\boldsymbol{n})_u\, dS, \tag{91}$$

where γ is the spanwise component of vorticity, v and $\boldsymbol{j}$ are the perturbation velocity component and the unit vector in the y-direction, respectively, and the angle θ is measured from the y-axis in the (y, z)-plane in the positive direction of rotation about the x-axis (figure 7). In arriving at (91), integrals of $(\partial\phi/\partial t)\,\boldsymbol{i}\cdot\boldsymbol{n}$ and $u(\boldsymbol{Q}\cdot\boldsymbol{n})$ around the edges of the wing and wake vortex sheets vanish.

Using (15),

$$\boldsymbol{n}_u = \frac{\partial h}{\partial x}\boldsymbol{i} + \frac{\partial h}{\partial y}\boldsymbol{j} - k + O(\epsilon^2), \quad \boldsymbol{n}_l = -\boldsymbol{n}_u, \tag{92}$$

and

$$\Delta\phi(x, y, t) = \int_{x_l(y)}^{x} \gamma(\xi, y, t)\, d\xi + O(\epsilon^2) \tag{93}$$

in (91) and integrating by parts in the next to the last term, we obtain

$$T(t) = \tfrac{1}{2}\rho \int_{S_1} (v^2 + w^2 - u^2)\, dS - \tfrac{1}{2}\rho \int_{S_3} (v^2 + w^2 - u^2)\, dS + \rho \int_{S_2} u(v\cos\theta + w\sin\theta)\, dS$$
$$+\rho \int_{-b}^{b} h_w(L(y), y, t) \frac{\partial}{\partial t}\Gamma(L(y), y, t)\, dy - \rho \int_{-b}^{b} dy \int_{x_l(y)}^{L(y)} dx \frac{\partial}{\partial t}[\gamma(x, y, t)\, h(x, y, t)]. \tag{94}$$

Here,

$$\Gamma(L(y), y, t) = \Gamma(y, t) + \int_{x_t(y)}^{L(y)} \gamma_w(\xi, y, t)\, d\xi, \tag{95}$$

where x_l, x_t and L are, respectively, the abscissae of the wing leading and trailing edges and that of the wake trailing edge. It can be shown that, as the far boundary S is removed to infinity, the integrals over S_1 and S_2 vanish. The integral over S_3 in general must be retained since, in the long-time limit, the wake usually crosses S_3 and the integral is expected to be non-zero. Equation (94) then reduces to

$$T(t) = -\tfrac{1}{2}\rho \int_{S_3} (v^2+w^2-u^2)\,\mathrm{d}S + \rho \int_{-b}^{b} h_w(L(y),y,t)\frac{\partial}{\partial t}\Gamma(L(y),y,t)\,\mathrm{d}y$$
$$-\rho\int_{-b}^{b}\mathrm{d}y\int_{x_l(y)}^{L(y)}\mathrm{d}x\frac{\partial}{\partial t}[\gamma(x,y,t)\,h(x,y,t)]. \quad (96)$$

This result is quite general, being valid for arbitrary small-amplitude transverse oscillations of a lifting surface of arbitrary planform, aspect ratio and reduced frequency. It can be shown that, as the wing semi-span b tends to infinity (96) reduces to its two-dimensional counterpart, (20). Also, in the limit of steady flow, the classical result for induced drag is recovered.

For steady-state harmonic oscillations the wake extends far beyond S_3 $(L\rightarrow\infty)$. After Reissner (1947; Reissner's theory is also presented in Bisplinghoff *et al.* 1955), we define a reduced circulation function

$$\tilde{\Omega}(y) = \frac{1}{c_0}\exp(j\bar{\omega}x_t(y))\,\tilde{\Gamma}(y), \quad (97)$$

where c_0 is the root semi-chord. Reissner shows that the spanwise component of wake vorticity is given by

$$\tilde{\gamma}_w(x,y) = -jk_0\,\tilde{\Omega}(y)\,\mathrm{e}^{-j\bar{\omega}x} \quad (x \geqslant x_t(y)), \quad (98)$$

where $k_0 = \omega c_0/U$ is the reduced frequency at the wing centre section.

Substituting (97) and (98) into (95), we obtain

$$\Gamma(L,y,t) = c_0\,\tilde{\Omega}(y)\,\mathrm{e}^{-j\bar{\omega}L}\,\mathrm{e}^{j\omega t}, \quad (99)$$

$$\frac{\partial}{\partial t}\Gamma(L,y,t) = -U\gamma_w(L,y,t). \quad (100)$$

Substituting this into (96), we obtain the thrust of a harmonically oscillating wing.

$$T(t) = -\tfrac{1}{2}\rho \int_{S_3} (v^2+w^2-u^2)\,\mathrm{d}S - \rho U \int_{-b}^{b} h_w(L,y,t)\,\gamma_w(L,y,t)\,\mathrm{d}y$$
$$-\rho\int_{-b}^{b}\mathrm{d}y\int_{x_l(y)}^{L}\mathrm{d}x\frac{\partial}{\partial t}[\gamma(x,y,t)\,h(x,y,t)]. \quad (101)$$

This is the three-dimensional counterpart of (27).

The average thrust is given by

$$\bar{T} = -\tfrac{1}{2}\rho \int_{S_3} (\overline{v^2}+\overline{w^2}-\overline{u^2})\,\mathrm{d}S - \rho U \int_{-b}^{b} \overline{h_w(L,y,t)\,\gamma_w(L,y,t)}\,\mathrm{d}y, \quad (102)$$

where both integrals are to be evaluated over the Trefftz plane. This is the three-dimensional counterpart of (28). Evaluation of the first integral in (102) requires knowledge of $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ in the far wake. These are calculated in Appendix C in terms of integrals involving $\tilde{\Omega}$ and $\mathrm{d}\tilde{\Omega}/\mathrm{d}y$ which in general must be evaluated numerically. Evaluation of the second integral in (102) requires the spanwise vorticity distribution, which is given by (98), and the displacement of the far wake.

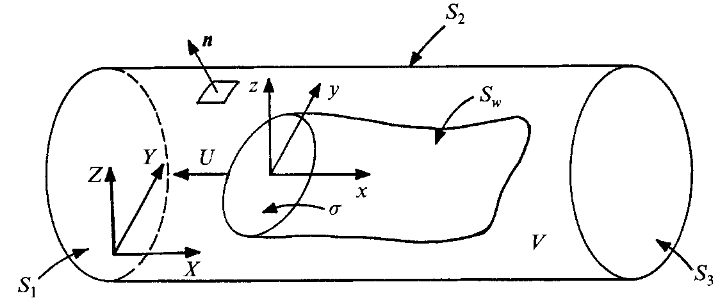

FIGURE 8. Control volume for conservation of energy in three dimensions.

For reasons already cited, we calculate the wake displacement from a planar wake model by inverting the linearized downwash at the wake. The result is

$$\tilde{h}_w(x,y) = \tilde{h}(x_t(y),y)\exp(-j\bar{\omega}(x-x_t(y))) + U^{-1}\int_{x_t(y)}^{x} \tilde{w}_w(\xi,y)\exp(-j\bar{\omega}(x-\xi))\,d\xi \qquad (x \geqslant x_t(y)). \tag{103}$$

The asymptotic value of $\tilde{h}$ is obtained from this by setting $x = \infty$ in the upper limit of the integral. Evaluation of (103) requires the downwash at the plane of the wake which is induced by the wing and wake vorticity. For (x,y) on the projection of the wake on the (x,y)-plane,

$$\tilde{w}_w(x,y) = \frac{-1}{4\pi}\int_{-b}^{b} d\eta \int_{x_l(y)}^{x_t(y)} d\xi \frac{(x-\xi)\,\tilde{\gamma}_a(\xi,\eta) + (y-\eta)\,\tilde{\delta}_a(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{3}{2}}} - \frac{1}{4\pi}\oint_{-b}^{b} d\eta \int_{x_t(y)}^{\infty} d\xi \frac{(x-\xi)\,\tilde{\gamma}_w(\xi,\eta) + (y-\eta)\,\tilde{\delta}_w(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{3}{2}}}, \tag{104}$$

where $(\,)_a$ denotes the wing and δ is the streamwise component of vorticity which is taken positive in the negative direction of rotation about the x-axis.

From Reissner (1947)

$$\tilde{\delta}_w(x,y) = c_0\,d\tilde{\Omega}/dy\,e^{-j\bar{\omega}x}. \tag{105}$$

It follows from (97) that

$$\tilde{\Omega}(y) = \frac{1}{c_0}\exp(j\bar{\omega}x_t(y))\int_{x_l(y)}^{x_t(y)} \tilde{\gamma}_a(\xi,y)\,d\xi. \tag{106}$$

δ_a is obtained from the continuity of vorticity on the wing.

$$\tilde{\delta}_a(x,y) = \frac{\partial}{\partial y}\int_{x_l(y)}^{x} \tilde{\gamma}_a(\xi,y)\,d\xi. \tag{107}$$

Hence, once the bound vorticity γ_a is determined, everything else can be determined. In the absence of an exact theory to calculate γ_a, one must use a numerical or approximate unsteady lifting-surface theory, many of which are available.

Next, we calculate the wake energy.

3.2. *Wake energy*

Again, we adopt the viewpoint of the observer fixed in the fluid and consider a lifting surface moving with velocity U along a rectilinear path in the negative X-direction while executing small-amplitude transverse oscillations, as shown in figure 8. The balance of energy for the fluid in V is given by (51) where the cylindrical far boundary S is located infinitely far from the wing and wake.

As in two dimensions, it can be shown that the right-hand side of (51) is the rate of work of total thrust T and unsteady lift, i.e.

$$\int_V \frac{\partial}{\partial t}(\tfrac{1}{2}\rho|\boldsymbol{q}|^2)\,\mathrm{d}V = -UT - \int_{S_a} \Delta p \frac{\partial h_a}{\partial t}\,\mathrm{d}S, \tag{108}$$

where S_a is the wing planform area. Taking the time average of (108) over the interval τ and rearranging, we obtain

$$-\int_{S_a} \overline{\Delta p \frac{\partial}{\partial t} h_a}\,\mathrm{d}S = U\bar{T} + \frac{1}{\tau}\Delta(KE), \tag{109}$$

where $1/\tau\,\Delta(KE)$ is defined in (59). This is a statement of conservation of energy for the present problem. Denoting the average total power required and energy loss rate, respectively, by $\bar{P}$ and $\bar{E}$, (109) becomes

$$\bar{P} = U\bar{T} + \bar{E} \quad \text{or} \quad C_P = C_T + C_E, \tag{110}$$

where the non-dimensional coefficients are defined as

$$C_P = \bar{P}/[\tfrac{1}{4}\pi\rho U^3(\tfrac{1}{2}S_a)], \quad C_T = \bar{T}/[\tfrac{1}{4}\pi\rho U^2(\tfrac{1}{2}S_a)], \quad C_E = \bar{E}/[\tfrac{1}{4}\pi\rho U^3(\tfrac{1}{2}S_a)]. \tag{111}$$

The hydrodynamic efficiency is defined in (62).

For harmonic oscillations we choose τ to be the period $2\pi/\omega$. As in two dimensions, $\Delta(KE)$ can be determined from the properties of the far wake, namely

$$\Delta(KE) = -\tfrac{1}{2}\rho\lambda \int_{-b}^{b} \overline{\overline{\Delta\phi\,\partial\phi/\partial z}}\bigg|_{Z=0}\,\mathrm{d}\eta. \tag{112}$$

$\Delta\phi$ is determined, with respect to the moving frame, using (93), (97) and (98).

$$\Delta\tilde{\phi}(x,y) = c_0\,\tilde{\Omega}(y)\,\epsilon^{-j\bar{\omega}x} \quad (x \geqslant x_t(y)). \tag{113}$$

Downwash at the plane of the wake is obtained from (C 6), as $z \to 0\pm$.

$$\frac{\partial\tilde{\phi}}{\partial z}(x,y,0) = -\frac{k_0}{2\pi}\mathrm{e}^{-j\bar{\omega}x} \oint_{-b}^{b} \frac{\mathrm{d}}{\mathrm{d}\eta}\tilde{\Omega}(\eta)\,\mathrm{sgn}\,(y-\eta)\,K_1(\bar{\omega}|y-\eta|)\,\mathrm{d}\eta$$
$$-\frac{k_0^2}{2\pi c_0}\mathrm{e}^{-j\bar{\omega}x} \int_{-b}^{b} \tilde{\Omega}(\eta)\,K_0(\bar{\omega}|y-\eta|)\,\mathrm{d}\eta, \tag{114}$$

where the K_n are modified Bessel functions of the second kind of order n and the sgn (z) function is -1 for $z < 0$ and 1 for $z > 1$. In the limit of steady flow, (113) and (114) reduce to the classical steady results.

The above expressions for $\Delta\phi$ and $\partial\Phi/\partial z$ are transformed to the (X, Y, Z) frame to obtain

$$\Delta\phi(X+Ut, Y) = c_0\,\tilde{\Omega}(y)\,\mathrm{e}^{-j\bar{\omega}X}, \tag{115}$$

$$\frac{\partial}{\partial z}\phi(X+Ut, Y)\bigg|_{Z=0} = -\frac{k_0}{2\pi}\mathrm{e}^{-j\bar{\omega}X} \oint_{-b}^{b} \frac{\mathrm{d}}{\mathrm{d}\eta}\tilde{\Omega}(\eta)\,\mathrm{sgn}\,(y-\eta)\,K_1(\bar{\omega}|y-\eta|)\,\mathrm{d}\eta$$
$$-\frac{k_0^2}{2\pi c_0}\mathrm{e}^{-j\bar{\omega}X} \int_{-b}^{b} \tilde{\Omega}(\eta)\,K_0(\bar{\omega}|y-\eta|)\,\mathrm{d}n. \tag{116}$$

The average energy loss rate is obtained from (115) and (116).

$$\bar{E} = \frac{1}{8\pi}\rho U k_0\,R_j\bigg\{c_0 \oint_{-b}^{b}\oint_{-b}^{b} \tilde{\Omega}^*(\eta_1)\frac{\mathrm{d}}{\mathrm{d}\eta}\tilde{\Omega}(\eta)\,\mathrm{sgn}\,(\eta_1-\eta)\,K_1(\bar{\omega}|\eta_1-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\eta_1$$
$$+k_0\int_{-b}^{b}\int_{-b}^{b} \tilde{\Omega}^*(\eta_1)\,\tilde{\Omega}(\eta)\,K_0(\bar{\omega}|\eta_1-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\eta_1\bigg\}. \tag{117}$$

The average total power required to maintain the wing oscillations is given by

$$\bar{P} = -\int_{-b}^{b} d\eta \int_{x_l(y)}^{x_t(y)} d\xi \, \overline{\Delta p(\xi, \eta, t) \frac{\partial}{\partial t} h_a(\xi, \eta, t)}. \tag{118}$$

The total thrust is then obtained from the balance of energy, (110). The present method requires $\tilde{\Omega}$ which must be obtained from unsteady lifting-surface theory, as pointed out above.

In the limit of steady flow, the above result for $\bar{E}$ yields half of the known induced drag. The factor of a half arises from time averaging which is not relevant in steady flow.

4. Summary

An alternative approach based on integral conservation laws has been presented for the calculation of the total energetic quantities for two- and three-dimensional lifting surfaces oscillating in inviscid incompressible flow. The method is applicable to lifting surfaces of arbitrary planform, aspect ratio, mode of oscillation (small amplitudes), and reduced frequency; and does not require calculation of the leading-edge suction force, whose accurate evaluation is usually difficult in three dimensions.

The analysis is carried out for arbitrary and harmonic transverse oscillations. In two dimensions, the latter results are obtained in closed form. In three dimensions, the vorticity distribution on the lifting surface is required as input to the calculations. Hence, unsteady lifting-surface theory must be employed as well.

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Appendix A. Evaluation of integrals

The following integrals arise in the two-dimensional analysis.

I. The integral

$$I_1 = \int_{-c}^{c} \left[\frac{c-x}{c+x}\right]^{\frac{1}{2}} \frac{dx}{(x-\lambda)(\xi-x)} \quad (\xi, \lambda \geqslant c) \tag{A 1}$$

can be evaluated from the contour integral

$$\oint_C \left[\frac{\zeta-c}{\zeta+c}\right]^{\frac{1}{2}} \frac{d\zeta}{(\zeta-\lambda)(\zeta-\xi)} = 2\pi i \left[\operatorname{Res}(\lambda) + \operatorname{Res}(\xi)\right] \quad (\xi, \lambda \geqslant c), \tag{A 2}$$

where $\zeta = x + iz$ and the integration contour is shown in figure 9. The integrand is defined with a branch cut from $\zeta = -c$ to $\zeta = c$. The result is

$$I_1 = \frac{-\pi}{\lambda-\xi} \left\{ \left[\frac{\lambda-c}{\lambda+c}\right]^{\frac{1}{2}} - \left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}} \right\} \quad (\lambda, \xi \geqslant c). \tag{A 3}$$

II. The integral

$$I_2 = \oint_{-c}^{c} \left[\frac{c-x}{c+x}\right]^{\frac{1}{2}} \frac{dx}{(x-\lambda)(\xi-x)} \quad (|\lambda| \leqslant c, \xi \geqslant c) \tag{A 4}$$

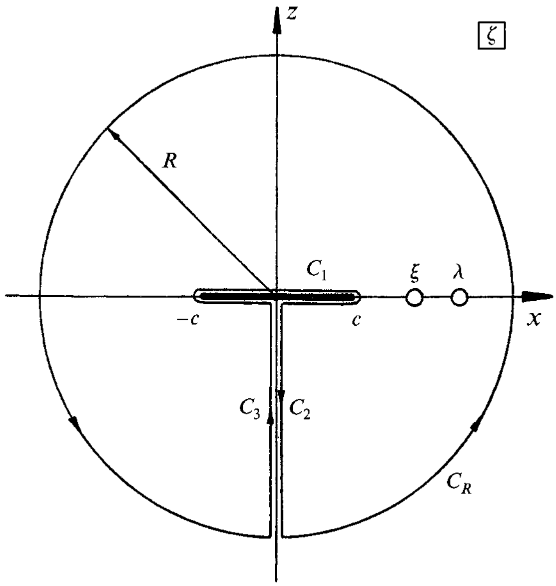

FIGURE 9. Integration contour for (A 2).

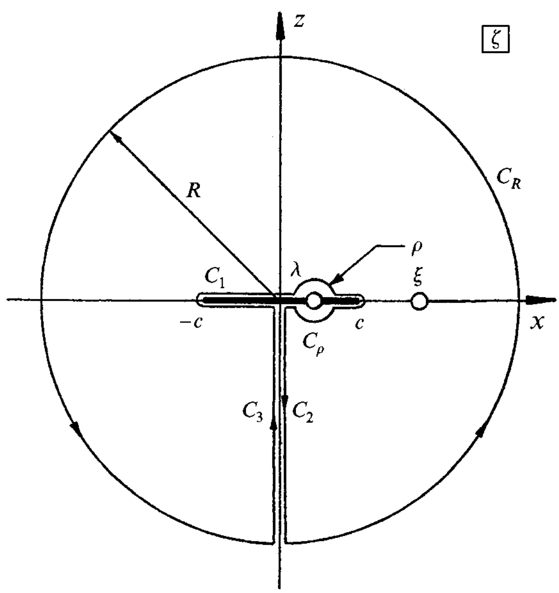

FIGURE 10. Integration contour for (A 5).

can be evaluated from the contour integral

$$\oint_C \left[\frac{\zeta-c}{\zeta+c}\right]^{\frac{1}{2}} \frac{d\zeta}{(\zeta-\lambda)(\zeta-\xi)} = 2\pi i \operatorname{Res}(\xi) \quad (|\lambda| \leqslant c, \quad \xi \geqslant c). \tag{A 5}$$

Again the integrand is defined with a branch cut from $\zeta = -c$ to $\zeta = c$ and the contour of integration is shown in figure 10. The result is

$$I_2 = \frac{\pi}{\lambda-\xi}\left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}} \quad (|\lambda| \leqslant c, \quad \xi \geqslant c). \tag{A 6}$$

III. The double integral

$$I_3 = \int_c^\infty d\xi \oint_c^\infty d\lambda \left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}} \left[\frac{\lambda+c}{\lambda-c}\right]^{\frac{1}{2}} \frac{e^{j\bar{\omega}(\xi-\lambda)}}{\xi-\lambda} \tag{A 7}$$

can be evaluated after differentiation with respect to $k = \bar{\omega}c$ which uncouples the integrals. The resulting integrals can be expressed in terms of certain known integrals (Ashley & Landahl 1965):

$$\int_c^\infty \left[\frac{\xi-c}{\xi+c}\right]^{\frac{1}{2}} e^{j\bar{\omega}\xi} d\xi = -\tfrac{1}{2}\pi c[H_1^{(2)*}(k) + jH_0^{(2)*}(k)], \tag{A 8}$$

$$\int_c^\infty \left[\frac{\lambda+c}{\lambda-c}\right]^{\frac{1}{2}} e^{-j\bar{\omega}\lambda} d\lambda = -\tfrac{1}{2}\pi c[H_1^{(2)}(k) + jH_0^{(2)}(k)], \tag{A 9}$$

where ( )* denotes complex conjugate. Substituting these results into (A 7), expressing the Hankel functions in terms of Bessel functions and integrating with respect to k, we obtain

$$I_3 = (\tfrac{1}{2}\pi)^2 c\left[J_0^2 + Y_0^2 - J_0^2(0) - Y_0^2(0) + j\int_0^k (J_1^2 + Y_1^2 - J_0^2 - Y_0^2)\, dk_1 + C'\right]. \tag{A 10}$$

The constant of integration C' is determined by direct evaluation of I_3, (A 7), for some value of $\bar{\omega}$ (or k). Using the method of stationary phase (Carrier, Krook & Pearson 1966), it can be shown that I_3 tends to zero as $\bar{\omega}$ (or k) $\rightarrow \infty$. Using this condition in (A 10), we find

$$I_3 = (\tfrac{1}{2}\pi)^2 c\left\{J_0^2(k) + Y_0^2(k) - j\int_k^\infty [J_1^2 + Y_1^2 - J_0^2 - Y_0^2]\, dk_1\right\}. \tag{A 11}$$

We do not need to evaluate the remaining integral here since it does not enter the expression for average thrust (see (43)).

Appendix B. Far wake perturbation velocities in two dimension

Here, we calculate the perturbation velocities in the far wake for a harmonically oscillating airfoil. Consider a two-dimensional wake extending infinitely far upstream and downstream of the Trefftz plane. Since u and w appear only in quadratic form in the expression for thrust, it suffices to calculate them from a linearized wake model. The strength of the wake vorticity is given by (see (22))

$$\tilde{\gamma}_w(x_1) = -jk\tilde{\Omega}\exp(-j\bar{\omega}x_1).$$

The Cartesian coordinate system (x_1, z) is attached to the plane of the wake at the Trefftz plane ($x_1 = 0$) and stationary with respect to (x, z).

The perturbation velocity potential in the far wake is given by

$$\tilde{\phi}(x_1, z) = \frac{jk\tilde{\Omega}}{2\pi}\int_{-\infty}^\infty e^{-j\bar{\omega}\xi} \tan^{-1}\left(\frac{z}{x_1-\xi}\right) d\xi. \tag{B 1}$$

Making the change of variables $\eta = x_1 - \xi$ and integrating by parts once, we obtain

$$\tilde{\phi}(x_1, z) = \frac{jk\tilde{\Omega}}{2\pi} z \frac{\exp(-j\bar{\omega}x_1)}{j\bar{\omega}} \int_{-\infty}^\infty \frac{\exp(j\bar{\omega}\eta)}{z^2+\eta^2} d\eta. \tag{B 2}$$

The imaginary part of this integral is identically zero, owing to symmetry. The real part is given by (Dwight 1961)

$$2\int_0^\infty \frac{\cos(\bar{\omega}\eta)}{z^2+\eta^2}\,\mathrm{d}\eta = \frac{\pi}{|z|}\mathrm{e}^{-\bar{\omega}|z|} \quad (z \neq 0). \tag{B 3}$$

The perturbation potential then becomes

$$\tilde{\phi}(x_1, z) = c\tilde{\Omega}\exp(-j\bar{\omega}x_1)\exp(-\bar{\omega}|z|)\,\mathrm{sgn}(z) \quad (z \neq 0). \tag{B 4}$$

The perturbation velocity components in the far wake are obtained from (B 4) by differentiation.

$$\tilde{u}(x_1, z) = = -\tfrac{1}{2}jk\tilde{\Omega}\exp(-j\bar{\omega}x_1)\exp(-\bar{\omega}|z|)\,\mathrm{sgn}(z) \quad (\mathrm{z} \neq 0), \tag{B 5}$$

$$\tilde{w}(x_1, z) = = -\tfrac{1}{2}k\tilde{\Omega}\exp(-j\bar{\omega}x_1)\exp(-\bar{\omega}|z|) \quad (z \neq 0). \tag{B 6}$$

As a check we note that, as $z \rightarrow 0\pm$,

$$\tilde{u} = \mp\tfrac{1}{2}jk\tilde{\Omega}\exp(-j\bar{\omega}x_1),$$
$$\tilde{w} = -\tfrac{1}{2}k\tilde{\Omega}\exp(-j\bar{\omega}x_1), \tag{B 7}$$

which are consistent with the symmetry properties of vortex sheets in unsteady motion.

The average of the square of the perturbation velocities in the far wake are given by

$$\overline{u^2}(x_1, z) = \overline{w^2}(x_1, z) = \tfrac{1}{8}k^2|\tilde{\Omega}|^2\exp(-2\bar{\omega}|z|). \tag{B 8}$$

Thus, in the far wake with respect to the body frame, u and w vary sinusoidally with x, but $\overline{u^2}$ and $\overline{w^2}$ are independent of x.

Appendix C. Far wake perturbation velocities in three dimensions

Here, we calculate the perturbation velocities in the far wake of a harmonically oscillating wing of finite span. Again, it suffices to calculate u, v and w from a linearized wake model. We consider a wake extending infinitely far upstream and downstream of the Trefftz plane and choose a Cartesian coordinate system (x_1, y, z) which is stationary in the (x, y, z) frame. The Trefftz plane coincides with the (y, z)-plane and the wake is defined by $|y| \leqslant b, z = 0$. The components of wake velocity are given by (98) and (105).

It follows from the Biot-Savart law and (98) that u is given by

$$\tilde{u}(\mathrm{x}_1, \mathrm{y}, \mathrm{z}) = -\frac{1}{4\pi}jk_0 z\int_{-b}^{b}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\xi\,\tilde{\Omega}(\eta)\,\mathrm{e}^{-j\bar{\omega}\xi}R_1^{-3}, \tag{C 1}$$

where $R_1 = ((x_1-\xi)^2+(y-\eta)^2+z^2)^{\frac{1}{2}}$. The integral over ξ can be expressed in terms of modified Bessel function of the second kind K_1 (Abramowitz & Stegun 1970).

$$\tilde{u}(x_1, y, z) = -\frac{1}{2\pi c_0}jk_0^2 z\exp(-j\bar{\omega}x_1)\int_{-b}^{b}\tilde{\Omega}(\eta)\,R_2^{-1}K_1(\bar{\omega}R_2)\,\mathrm{d}\eta, \tag{C 2}$$

where $R_2 = ((y-\eta)^2+z^2)^{\frac{1}{2}}$. It follows from the Biot-Savart law and (105) that v is given by

$$\tilde{v}(x_1, y, z) = \frac{1}{4\pi}c_0 z\int_{-b}^{b}\mathrm{d}\eta\int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\mathrm{d}}{\mathrm{d}\eta}\tilde{\Omega}(\eta)\exp(-j\bar{\omega}\xi)\,R_1^{-3}. \tag{C 3}$$

The integral over ξ can be expressed in terms of K_1.

$$\tilde{v}(x_1, y, z) = \frac{1}{2\pi}k_0 z\exp(-j\bar{\omega}x_1)\int_{-b}^{b}\frac{\mathrm{d}}{\mathrm{d}\eta}\tilde{\Omega}(\eta)\,R_2^{-1}K_1(\bar{\omega}R_2)\,\mathrm{d}\eta. \tag{C 4}$$

Using the asymptotic expansion for K_1, it can be shown that as $z \to 0\pm$, the integrals in (C 2) and (C 4) each contain a second-order singularity, $(y-\eta)^{-2}$, and must be interpreted according to the principle value of Mangler (1951).

It follows from the Biot-Savart law and (98) and (105) that w is given by

$$\tilde{w}(x_1, y, z) = -\frac{1}{4\pi} c_0 \int_{-b}^{b} d\eta \int_{-\infty}^{\infty} d\xi \,(y-\eta) \frac{d}{d\eta} \tilde{\Omega}(\eta) \exp(-j\bar{\omega}\xi)\, R_1^{-3}$$
$$+ \frac{1}{4\pi} jk_0 \int_{-b}^{b} d\eta \int_{-\infty}^{\infty} d\xi (x_1 - \xi)\, \tilde{\Omega}(\eta) \exp(-j\bar{\omega}\xi)\, R_1^{-3}. \quad (C\ 5)$$

The integral over ξ in the first term can be expressed in terms of K_1 and the one in the second term can be expressed in terms of K_0 (Abramowitz & Stegun 1970).

$$\tilde{w}(x_1, y, z) = -\frac{1}{2\pi} k_0 \exp(-j\bar{\omega}x_1) \int_{-b}^{b} (y-\eta) \frac{d}{d\eta} \tilde{\Omega}(\eta)\, R_2^{-1} K_1(\bar{\omega}R_2)\, d\eta$$
$$- \frac{1}{2\pi c_0} k_0^2 \exp(-j\bar{\omega}x_1) \int_{-b}^{b} \tilde{\Omega}(\eta)\, K_0(\bar{\omega}R_2)\, d\eta. \quad (C\ 6)$$

Using the asymptotic expansion for K_1, it can be shown that as $z \to 0\pm$, the integral in (C 6) containing K_1 has a first-order singularity, $(y-\eta)^{-1}$, and must be interpreted as a Cauchy principal value.

The above results show that in the far wake u, v and w vary sinusoidally with x, but $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$ are independent of x. In the limit of steady flow, the above expressions for u, v and w reduce to their known steady values.

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